

Comments on “On the Consistency of the Solutions of the Space
Fractional Schrödinger Equation” [J. Math. Phys. **53**, 042105
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Abstract

Recently we have reanalyzed the consistency of the solutions of the space fractional Schrödinger equation found in a piecewise manner, and showed that an exact and a proper treatment of the relevant integrals prove that they are consistent. In this comment, for clarity, we present additional information about the critical integrals and describe how their analytic continuation is accomplished. [doi: 10.1063/1.4705268]

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In a recent article [1], we have reanalyzed the consistency problem of the solutions of the space fractional Schrödinger equation found in a piecewise fashion and showed that a proper treatment of the relevant integrals, in contrast to the claims of Jeng et.al. [2], proves that they are consistent. A crucial step of our proof was the analytic continuation of certain integrals in evaluating their Cauchy principal values. In order to remove any question marks about how analytic continuation is accomplished, we present some additional details.

The space fractional Schrödinger equation is written by using the Riesz derivative, R_x^α , which is defined with respect to the Fourier transform

$$\mathcal{F}\{R_x^\alpha(g(x))\} = -\frac{[(iq)^\alpha + (-iq)^\alpha]}{2\cos(\alpha\pi/2)}G(q), \quad 1 < \alpha \leq 2, \quad (1)$$

where $\mathcal{F}\{g(x)\} = G(q)$. Taking the inverse, gives the Riesz derivative as [3,4]

$$R_x^\alpha(g(x)) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{[(iq)^\alpha + (-iq)^\alpha]}{2\cos(\alpha\pi/2)} G(q) e^{iqx} dq. \quad (2)$$

Since in equation (2) q is real, using the relation

$$[(iq)^\alpha + (-iq)^\alpha] = |q|^\alpha 2\cos(\alpha\pi/2), \quad (3)$$

it is customary to write the Riesz derivative as

$$R_x^\alpha(g(t)) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} |q|^\alpha G(q) e^{i\omega x} dq. \quad (4)$$

In this regard, Equation (12) of the paper [1] was written as

$$\psi_1(x) = -\frac{AD_\alpha}{\pi E_1} \left(\frac{\pi\hbar}{2a}\right)^\alpha \int_{-\infty}^{+\infty} dq \frac{|q|^\alpha \cos(\pi q/2)}{q^2 - 1} e^{i\pi q x/2a}, \quad (5)$$

where q is real.

To evaluate the Cauchy principal value of the above integral, we make use of the Cauchy integral theorem, which requires that the integrand be analytic in and on the closed contour described in [1]. Since $|q|^\alpha$ is not analytic in the complex q -plane, the proper analytic continuation of this integral is accomplished by going to the original form of the Riesz derivative [Eq. (2)], and by writing Equation (5) with the replacement

$$|q|^\alpha \rightarrow \frac{[(iq)^\alpha + (-iq)^\alpha]}{2\cos(\alpha\pi/2)}, \quad (6)$$

and then performing analytic continuation. Similarly, analytic continuation of Equation (25) of the paper is performed.

It is important to note that Equation (9) of [1]:

$$\psi_1(x) = \begin{cases} A \cos\left(\frac{\pi x}{2a}\right) & \text{if } |x| < a \\ 0 & \text{if } |x| \geq a \end{cases} . \quad (7)$$

and equation (5), which are claimed to be inconsistent [2], are basically the same wave function. Equation (5) is just the integral representation of the wave function in Equation (7). We showed that when the integral is evaluated properly, they indeed agree with each other [1]. This is true not just for the ground state but for all the other states and also in general.

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